An Introduction to Variational Inequalities

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1 Introduction and motivation

2 Variational Inequalities in $\mathbb{R}^N$

3 Variational Inequalities in Hilbert spaces

4 The obstacle problem
Consider a Nash-Equilibrium problem:

- Business $i = 1, \ldots, n$ produce the same product.
- $\xi = \sum_{i=1}^{n} x_i$ total units of the produced product.
- $p(\xi)$ the price (per unit) the consumer asks the units of the product.
- $c_i(x_i)$ costs for the production of $x_i$ units of the company $i$.

**Nash-Equilibrium**

A $x^* \in \mathbb{R}^n$ is called a Nash-Equilibrium, if the Problem

$$\begin{cases}
\max \left( x_i \cdot p \left( x_i + \sum_{i \neq j} x_j^* \right) - c_i(x_i) \right) \\
x_i \geq 0
\end{cases}$$

is solved, for all $x_i^*, i = 1, \ldots, n$. 
An introductory problem

\[ \xi = \sum_{i=1}^{n} x_i \ldots \text{total units}; \quad p(\xi) \ldots \text{the price}; \quad c_i(x_i) \ldots \text{costs} \]

In short notation: \[ g_i(x_i) = x_i \cdot p \left( x_i + \sum_{i \neq j} x_j^* \right) - c_i(x_i) \]

Nash-Equilibrium

\[ \begin{cases} 
\max g_i(x_i) \\
 x_i \geq 0 
\end{cases} \]

Let \( g \) be differentiable, then a necessary condition for the solution \( x^* \) of the Problem is:

\[ g_i'(x_i^*)(x_i - x_i^*) \leq 0. \]
An introductory problem

\[ \xi = \sum_{i=1}^{n} x_i \ldots \text{total units; } p(\xi) \ldots \text{the price; } c_i(x_i) \ldots \text{costs} \]

Assume:

- \( \exists x_i \geq 0: g'_i(x_i^*)(x_i - x_i^*) > 0 \)
- Consider: \( t \in (0, 1): x_i^* + t(x_i - x_i^*) = (1 - t)x_i^* + tx_i \geq 0 \)

Taylor series

\[ g_i(x_i^* + t(x_i - x_i^*)) = g_i(x_i^*) + g'_i(x_i^*)t(x_i - x_i^*) + o(t^2(x_i - x_i^*)) \]
**An introductory problem**

\[ \xi = \sum_{i=1}^{n} x_i \quad \text{...total units; } p(\xi) \quad \text{...the price; } c_i(x_i) \quad \text{...costs} \]

Assume:

- \( \exists x_i \geq 0 : g'_i(x_i^*)(x_i - x_i^*) > 0 \)
- Consider: \( t \in (0, 1) : x_i^* + t(x_i - x_i^*) = (1-t)x_i^* + tx_i \geq 0 \)

Taylor series

\[ g_i(x_i^* + t(x_i - x_i^*)) = g_i(x_i^*) + g'_i(x_i^*)t(x_i - x_i^*) + o(t^2(x_i - x_i^*)) \]

\( t \text{ sufficiently small} \)

\[ \implies g_i(x_i^* + t(x_i - x_i^*)) > g_i(x_i^*) \]

**Contradiction (to } x_i^* \text{ is solution of the Problem.)} \)
AN INTRODUCTORY PROBLEM

\[ \xi = \sum_{i=1}^{n} x_i \ldots \text{total units; } p(\xi) \ldots \text{the price; } c_i(x_i) \ldots \text{costs} \]

Set \( F(x) = \left[ c'_i(x_i) - p(\xi) - x_i \cdot p'(\xi) \right]_{i=1}^{n} \).

**Variational Inequality**

Find \( x^* \in \mathbb{R}^n, x^* \geq 0 \) such that

\[ F(x^*)(x - x^*) \geq 0 \]

holds for all \( x \in \mathbb{R}^n, x \geq 0 \).
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PROJECTION

LEMMA

Let $\mathcal{K}$ be a closed, convex subset of a Hilbert space $H$. Then for each $x \in H$ there is a unique $y \in \mathcal{K}$ such that

$$\|x - y\| = \inf_{\eta \in \mathcal{K}} \|x - \eta\|.$$ 

A $y \in \mathcal{K}$ which satisfy this equation is called the projection of $x$ on $\mathcal{K}$ and we write

$$y = \text{Pr}_\mathcal{K}(x).$$

THEOREM

Let $\mathcal{K}$ be a closed, convex subset of a Hilbert space $H$. Then for $x \in H$

$$y = \text{Pr}_\mathcal{K}(x) \iff y \in \mathcal{K} \text{ and } (\forall \eta \in \mathcal{K}) : (y, \eta - y) \geq (x, \eta - y).$$
**PROJECTION**

**Lemma**

Let $\mathcal{K}$ be a closed, convex subset of a Hilbert space $H$. Then for each $x \in H$ there is a unique $y \in \mathcal{K}$ such that

$$
\|x - y\| = \inf_{\eta \in \mathcal{K}} \|x - \eta\|.
$$

A $y \in \mathcal{K}$ which satisfy this equation is called the **projection of** $x$ **on** $\mathcal{K}$ and we write

$$
y = \text{Pr}_\mathcal{K}(x).
$$

**Theorem**

Let $\mathcal{K}$ be a closed, convex subset of a Hilbert space $H$. Then for $x \in H$

$$
y = \text{Pr}_\mathcal{K}(x) \iff y \in \mathcal{K} \text{ and } (\forall \eta \in \mathcal{K}) : (y, \eta - y) \geq (x, \eta - y).
$$
**THEOREM**

Let $\mathcal{K} \subset \mathbb{R}^N$ be compact, convex and let $F : \mathcal{K} \rightarrow (\mathbb{R}^N)'$ be continuous. Then exist $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathcal{K}$$

holds.

**PROOF.**

The mapping $Pr_{\mathcal{K}}(id - \pi F) : \mathcal{K} \rightarrow \mathcal{K}$ is continuous.

Fix-point theorem of Brouwer gives: $\exists x \in \mathcal{K} : x = Pr_{\mathcal{K}}(id - \pi F)(x)$
A FIRST THEOREM

**Theorem**

Let $\mathcal{K} \subset \mathbb{R}^N$ be compact, convex and let $F : \mathcal{K} \rightarrow (\mathbb{R}^N)'$ be continuous. Then exist $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \ \forall y \in \mathcal{K}$$

holds.

**Proof.**

That gives for all $y \in \mathcal{K}$

$$(x, y - x) \geq ((id - \pi F)x, y - x) = (x, y - x) - (\pi F(x), y - x)$$
Theorem

Let $\mathcal{K} \subset \mathbb{R}^N$ be compact, convex and let $F : \mathcal{K} \to (\mathbb{R}^N)'$ be continuous. Then exist $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathcal{K}$$

holds.

Proof.

Thus we finally obtain

$$\left( \forall y \in \mathcal{K} \right) : \langle F(x), y - x \rangle = (\pi F(x), y - x) \geq 0.$$
Problem 1

Let $\mathcal{K} \subset \mathbb{R}^N$ be closed, convex and let $F : \mathcal{K} \rightarrow (\mathbb{R}^N)'$ continuous. Find a $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \text{ for all } y \in \mathcal{K}$$
Criteria for solutions in the $\mathbb{R}^N$

**Problem 1**

Let $\mathcal{K} \subset \mathbb{R}^N$ be closed, convex and let $F : \mathcal{K} \to (\mathbb{R}^N)'$ continuous. Find a $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \text{ for all } y \in \mathcal{K}$$

We denote

$$\mathcal{K}_R = \mathcal{K} \cap \overline{B_R}(0)$$

**Theorem**

Let $\mathcal{K} \subset \mathbb{R}^N$ be closed, convex and let $F : \mathcal{K} \to (\mathbb{R}^N)'$ be continuous. A necessary and sufficient condition for the existence of a solution to problem 1 is:

It exists a $R > 0$ such that a solution $x_R \in \mathcal{K}_R$ of

$$\langle F(x_R), y - x_R \rangle \geq 0 \text{ for all } y \in \mathcal{K}_R$$

satisfies $\|x_R\| < R$. 

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Criteria for solutions in the $\mathbb{R}^N$

Problem 1

Let $\mathcal{K} \subset \mathbb{R}^N$ be closed, convex and let $F : \mathcal{K} \to (\mathbb{R}^N)'$ continuous. Find a $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \text{for all } y \in \mathcal{K}$$

Proof.

Necessary: If $x \in \mathcal{K}$ solves the Problem 1, you can obviously find a $R > 0$ such that

$$\|x_R\| < R \text{ and } x_R \in \mathcal{K}_R$$

holds.
Criteria for solutions in the $\mathbb{R}^N$

**Problem 1**

Let $\mathcal{K} \subset \mathbb{R}^N$ be closed, convex and let $F : \mathcal{K} \to (\mathbb{R}^N)'$ continuous. Find a $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \text{ for all } y \in \mathcal{K}$$

**Proof.**

**Sufficient:** Let $x_R \in \mathcal{K}_R$ with $\|x_R\| < R$, then exists a $\varepsilon > 0$ such that for all $y \in \mathcal{K}$, $w = x_R + \varepsilon(y - x_R) \in \mathcal{K}_R$ holds.

Thus we get

$$0 \leq \langle F(x_R), w - x_R \rangle = \varepsilon \langle F(x_R), y - x_R \rangle \ \forall y \in \mathcal{K}.$$
Criteria for solutions in the $\mathbb{R}^N$

**Problem 1**

Let $\mathcal{K} \subseteq \mathbb{R}^N$ be closed, convex and let $F : \mathcal{K} \rightarrow (\mathbb{R}^N)'$ continuous. Find a $x \in \mathcal{K}$ such that

$$\langle F(x), y - x \rangle \geq 0 \text{ for all } y \in \mathcal{K}$$

**Theorem**

Let $F : \mathcal{K} \rightarrow (\mathbb{R}^N)'$. If

$$\lim_{\|x\| \rightarrow +\infty} \frac{\langle F(x) - F(x_0), x - x_0 \rangle}{\|x - x_0\|} = +\infty$$

holds for a $x_0 \in \mathcal{K}_R$, then there exists a solution of Problem 1.
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Problem 2

Let $K \subset H$ be closed and convex and $f \in H'$. To find:

$$u \in K : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K.$$ 

- $H$ denotes a Hilbert space and $H'$ its dual space.
- The pairing between $H$ and $H'$:

$$\begin{align*}
\left\{ \begin{array}{c}
H' \times H \rightarrow \mathbb{R} \\
f, x \mapsto \langle f, x \rangle
\end{array} \right.
\end{align*}$$

- $a(u, v)$ a coercive bilinear form on $H$ with $\alpha$ as coercive constant.
- and let $a_t(u, v) := a_0(u, v) + tb(u, v)$ with
  - $a_0(u, v) := \frac{1}{2} (a(u, v) + a(v, u))$ the symmetric part,
  - $b(u, v) := \frac{1}{2} (a(u, v) - a(v, u))$ the antisymmetric part.
**Problem 2**

Let $\mathcal{K} \subset H$ be closed and convex and $f \in H'$. To find:

$$u \in \mathcal{K} : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.$$ 

- $H$ denotes a Hilbert space and $H'$ its dual space.
- The pairing between $H$ and $H'$:

$$\left\{ \begin{array}{c} H' \times H \rightarrow \mathbb{R} \\ f, x \mapsto \langle f, x \rangle \end{array} \right.$$ 

- $a(u, v)$ a coercive bilinear form on $H$ with $\alpha$ as coercive constant.
- and let $a_t(u, v) := a_0(u, v) + tb(u, v)$ with
  - $a_0(u, v) := \frac{1}{2} (a(u, v) + a(v, u))$ the symmetric part,
  - $b(u, v) := \frac{1}{2} (a(u, v) - a(v, u))$ the antisymmetric part.
Technical tools

Problem 2
Let $\mathcal{K} \subset H$ be closed and convex and $f \in H'$. To find:

$$u \in \mathcal{K} : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.$$ 

Lemma
Let $a(u, v)$ be a coercive bilinear form on $H$, $\mathcal{K} \subset H$ closed and convex and $f \in H'$. Then is the mapping $f \mapsto u$ Lipschitz, that is, if $u_1, u_2$ are solutions to problem 2 corresponding to $f_1, f_2 \in H'$, then

$$\|u_1 - u_2\| \leq \frac{1}{\alpha} \|f_1 - f_2\|.$$
**Problem 2**

Let $\mathcal{K} \subset H$ be closed and convex and $f \in H'$. To find:

$$u \in \mathcal{K} : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.$$ 

**Lemma**

Let $a(u, v)$ be a coercive bilinear form on $H$, $\mathcal{K} \subset H$ closed and convex and set

$$M = \sup_{u, v \in H} \frac{|b(u, v)|}{\|u\| \cdot \|v\|} < +\infty.$$ 

If Problem 2 is solvable for $a_\tau(u, v)$ and all $f \in H'$, it is also solvable for $a_t(u, v)$ and all $f \in H'$, with $\tau \leq t \leq \tau + t_0$ and $t_0 < \frac{\alpha}{M}$ and a unique solution $u$. 

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Problem 2

Let $K \subset H$ be closed and convex and $f \in H'$. To find:

$$u \in K : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K.$$ 

Proof.

Sketch: Define

$$T : \begin{cases} \quad H & \longrightarrow K \\ Tw & = u \end{cases}.$$

Define the functional $F_t$:

$$\langle F_t, v \rangle = \langle f, v \rangle - (t - \tau)b(w, v) \quad \text{for} \quad \tau \leq t \leq \tau + t_0$$

for all $f \in H'$.
Problem 2

Let $K \subset H$ be closed and convex and $f \in H'$. To find:

$$u \in K : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K.$$  

Proof.

Let $u \in K$ such that $a_T(u, v)$ solves Problem 2 for $F_t$.

$$a_T(u, v - u) \geq \langle F_t, v - u \rangle \quad (\forall v \in K).$$

Therefore is $T$ well define. Use the fixpoint theorem of Banach to find a unique solution.
Existence of a solution

Problem 2
Let \( K \subset H \) be closed and convex and \( f \in H' \). To find:

\[
u \in K : \ a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K.
\]

Theorem (Existence of a solution)
Let \( a(u, v) \) be a coercive bilinear form on \( H \), \( K \subset H \) closed and convex and \( f \in H' \). Then exists a unique solution to problem 2.
**Existence of a solution**

**Proof.**

Thirst step: Let \( a(u, v) \) be symmetric.

Define:

\[
I(u) =: a(u, u) - 2\langle f, u \rangle \quad (\forall u \in H)
\]

Now \( a \) is coercive, so it follows

\[
I(u) \geq -\frac{1}{\alpha} \|f\|_{H'}^2.
\]

Set \( d =: \inf_{u \in \mathcal{K}} I(u) \geq -\frac{1}{\alpha} \|f\|_{H'}^2 > -\infty \).

Choose a minimizing sequence \( \{u_n \in \mathcal{K} | d \leq I(u_n) \leq d + \frac{1}{n}\} \) \( n \in \mathbb{N} \).

So we get

\[
\alpha \|u_n - u_m\|^2 \leq 2 \left[ \frac{1}{n} + \frac{1}{m} \right].
\]
Existence of a solution

**Proof.**

So \( \{u_n\}_{n \in \mathbb{N}} \) is Cauchy. 
\( \mathcal{K} \) is closed, therefore it exists a \( u \in \mathcal{K} \) such that 

\[
I(u) = d
\]

Let \( v \in \mathcal{K} \) and \( 0 \leq \varepsilon \leq 1 \). 
The set \( \mathcal{K} \) is convex, so 

\[
(1 - \varepsilon)u + \varepsilon v = u + \varepsilon(v - u) \in \mathcal{K}
\]

\( u \) is minimizing so we get 

\[
0 \leq 2a(u, v - u) - 2\langle f, v - u \rangle
\]

and therefore 

\[
a(u, v - u) \geq \langle f, v - u \rangle \quad (\forall v \in \mathcal{K})
\]
Existence of a solution

**Proof.**

Second step: Consider \( a_t(u, v) = a_0(u, v) + tb(u, v) \). In the case of \( t = 0 \) is \( a_t \) symmetric. Moreover

\[
0 < M =: \sup \left| \frac{b(u, v)}{\|u\| \cdot \|v\|} \right| < +\infty
\]

So choose a \( 0 < t_0 \leq \frac{\alpha}{M} \) and the problem is solved (unique) with the previous lemma for all \( 0 \leq t \leq t_0 \).

With iterative application of this argument the problem is solvable for \( t = 1 \).

That means the problem is solved for \( a_1(u, v) = a(u, v) \).
The obstacle problem

The problem

Let $\psi$ be a function on $\overline{\Omega} = \Omega \cup \partial \Omega$ such that

$$\max_{\Omega} \psi \geq 0 \text{ and } \psi \leq 0 \text{ on } \partial \Omega$$

holds.

Define $\mathcal{K} = \{ v \in C^1(\overline{\Omega}) | v \geq \psi \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial \Omega \}$. Find a $u \in \mathcal{K}$ such that:

$$\int_{\Omega} |\nabla u|^2 \, dx = \min_{v \in \mathcal{K}} \int_{\Omega} |\nabla v|^2 \, dx$$

Assume such a $u$ exists, with the convexity of $\mathcal{K}$ it follows that

$$\Phi(t) = \int_{\Omega} |\nabla (u + t(v - u))|^2 \, dx \text{ with } 0 \leq t \leq 1$$

is minimized at $t = 0$. 
The problem

Let $\psi$ be a function on $\Omega = \Omega \cup \partial \Omega$ such that

$$\max_{\Omega} \psi \geq 0 \text{ and } \psi \leq 0 \text{ on } \partial \Omega$$

holds.

Define $\mathcal{K} = \{ v \in C^1(\overline{\Omega}) | v \geq \psi \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial \Omega \}$. Find a $u \in \mathcal{K}$ such that:

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Assume such a $u$ exists, with the convexity of $\mathcal{K}$ it follows that

$$\Phi(t) = \int_{\Omega} |\nabla (u + t(v - u))|^2 \, dx \text{ with } 0 \leq t \leq 1$$

is minimized at $t = 0$. That leads to

$$\int_{\Omega} \nabla u \nabla (v - u) \, dx \geq 0 \text{ for all } v \in \mathcal{K}.$$
**The problem**

Let $\psi$ be a function on $\overline{\Omega} = \Omega \cup \partial \Omega$ such that

$$\max_{\Omega} \psi \geq 0 \text{ and } \psi \leq 0 \text{ on } \partial \Omega$$

holds.

Define $\mathcal{K} = \{v \in C^1(\overline{\Omega}) | v \geq \psi \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial \Omega\}$. Find a $u \in \mathcal{K}$ such that:

$$\int_\Omega |\nabla u|^2 \, dx = \min_{v \in \mathcal{K}} \int_\Omega |\nabla v|^2 \, dx$$

Assume such a $u$ exists, with the convexity of $\mathcal{K}$ it follows that

$$\Phi(t) = \int_\Omega |\nabla (u + t(v - u))|^2 \, dx \text{ with } 0 \leq t \leq 1$$

is minimized at $t = 0$. That leads to

$$\int_\Omega \nabla u \nabla (v - u) \, dx \geq 0 \text{ for all } v \in \mathcal{K}. $$
The obstacle problem

First properties

Problem 3

Let $f \in H^{-1}(\Omega)$ given. To find:

$$u \in K : \ a(u, v - u) \geq \langle f, v - u \rangle \ \forall \ v \in K.$$ 

- Let $\Omega \subset \mathbb{R}^N$ be bounded, $\partial \Omega$ be continuous.
- Let $a_{ij} \in L^{\infty}(\Omega)$ such that for $\xi \in \mathbb{R}^N$ and for almost all $x \in \Omega$

$$\frac{1}{\Lambda} \xi^2 \leq a_{ij}(x)\xi_i \xi_j \leq \Lambda \xi^2$$

holds.
- Set

$$a(u, v) = \int_{\Omega} a_{ij}(x) u_{x_i} u_{x_j} \, dx \ \text{for} \ u, v \in H^1(\Omega).$$
**Problem 3**

Let \( f \in H^{-1}(\Omega) \) given. To find:

\[
 u \in \mathcal{K} : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.
\]

**Theorem**

There is a unique solution of the problem 3.

**Proof.**

It holds by assumption, that for all \( v \in \mathcal{K} \)

\[
a(v, v) = \int_{\Omega} a_{ij}(x)v_{x_i}v_{x_j} \, dx \geq \frac{1}{\Lambda} \|u\|_{H^1_0(\Omega)}^2.
\]

Therefore \( a \) is coercive, and thus the problem 3 has a solution.
**First properties**

**Problem 3**

Let $f \in H^{-1}(\Omega)$ given. To find:

$$u \in \mathcal{K} : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.$$ 

**Definition**

We say that $g \in H^1(\Omega)$ is a supersolution of $L - f$ if

$$\langle Lg - f, \zeta \rangle \equiv a(g, \zeta) - \langle f, \zeta \rangle \geq 0$$

for $0 \leq \zeta \in H^1_0(\Omega)$.

Reminder:

$$\mathcal{K} = \{ v \in C^1(\overline{\Omega}) | v \geq \psi \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial \Omega \}$$
**Problem 3**

Let \( f \in H^{-1}(\Omega) \) given. To find:

\[
\begin{align*}
    u & \in \mathcal{K} : \\
    a(u, v - u) & \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.
\end{align*}
\]

**Theorem (Characterize solutions of problem 3)**

Let \( u \) be the solution of problem 3 and suppose that \( g \) is a supersolution of \( L - f \) satisfying \( g \geq \psi \) in \( \Omega \) and \( g \geq 0 \) on \( \partial \Omega \). Then

\[
u \leq g \quad \text{in} \ \Omega.
\]
The Obstacle Problem in 1D

Define a bilinear form as

\[ a(u, v) = \int_{\alpha}^{\beta} u'(x)v'(x)dx \quad \text{for } u, v \in H^1(\Omega). \]

Consider:

\begin{itemize}
  \item \( \Omega = (\alpha, \beta) \) an open interval of \( \mathbb{R} \).
  \item Let \( \mathcal{K} = \{ v \in H^1_0(\Omega) | v \geq \psi \text{ in } \Omega \} \) with \( \psi \in H^1(\Omega) \) satisfying
    \[ \max_{\Omega} \psi > 0, \quad \psi(\alpha) < 0, \quad \psi(\beta) < 0. \]
\end{itemize}
Define a bilinear form as

\[ a(u, v) = \int_{\alpha}^{\beta} u'(x)v'(x)\,dx \quad \text{for } u, v \in H^1(\Omega). \]

Consider:

- \( \Omega = (\alpha, \beta) \) an open interval of \( \mathbb{R} \).
- Let \( \mathcal{K} = \{ v \in H^1_0(\Omega) | v \geq \psi \text{ in } \Omega \} \) with \( \psi \in H^1(\Omega) \) satisfying

\[
\max_{\Omega} \psi > 0, \quad \psi(\alpha) < 0, \quad \psi(\beta) < 0.
\]
**Problem 3**

Let \( f \in H^{-1}(\Omega) \) given. To find:

\[
  u \in \mathcal{K} : \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in \mathcal{K}.
\]

---

**Theorem**

Let \( f \in H^{-1}(\Omega) \) such that there is a continuous \( F \in L^2(\Omega) \) with \( f = F' \). If \( \psi'(x) \) has only discontinuities of the form

\[
  \psi'(x - 0) \leq \psi'(x + 0)
\]

then for the solution \( u \) of the problem holds \( u'(x) \) is continuous.
Master project: The obstacle problem

Motivation:
In flow assurance, the simulation of multi-phase flow in pipes is important. The profile of the pipe is a sensitive input parameter to the simulation. Close to the horizontal, fractions of degrees can change the flow in the pipes significantly.

(Left) Example of a pipelayer rigg. (Right) Image of a pipe on the seabed.
Often, only a echo sounding image of the ocean surface is available. This echo sounding image has a resolution of typically 1 meter in the horizontal direction and 1 cm in the vertical direction. The pipe is a stiff object which will only touch the seabed at some points.

(Left) Echo sounding image of the seabed. (Right) Sketch of a pipelayer rigg, and the pipe resting on parts of the seabed.

We want to be able to compute the profile of the pipe based on the echo image, and a stiffness description of the pipe.
For Further Reading

D. Kinderlehrer
AN INTRODUCTION TO VARIATIONAL INEQUALITIES AND THEIR APPLICATIONS.

F. Tröltzsch
Optimale Steuerung partieller Differentialgleichungen